Note on Operator Algebras

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Chapter 1

General Topology

Topology is an abstract structure that can be built on the set theory. We start with introducing the topological structure by open stets, which is the most standard way.

A topological space is a set Ω together with $\mathscr{O}$, a collection of subsets of Ω, satisfying the following properties:

- $\phi \in \mathscr{O}$ and $\Omega \in \mathscr{O}$.
- If $O_1 \in \mathscr{O}$ and $O_2 \in \mathscr{O}$, then $O_1 \cap O_2 \in \mathscr{O}$.
- If $O_\alpha \in \mathscr{O}$ $(\alpha \in I)$ for arbitrary set of suffixes, then $\cup_{\alpha \in I} O_\alpha \in \mathscr{O}$.

An element of $\mathscr{O}$ is called an open set.

In general, a set may have several topologies.

If two topologies satisfy $\mathscr{O}_1 \subset \mathscr{O}_2$, then $\mathscr{O}_1$ is called weaker than $\mathscr{O}_2$, or smaller than $\mathscr{O}_2$.

Topological structure can be generated by a subset of open spaces.

Let $\mathscr{B}$ be a collection of subsets of a set Ω. The weakest topology $\mathscr{O}$ such that $\mathscr{B} \subset \mathscr{O}$ is called generated by $\mathscr{B}$. We note that such $\mathscr{O}$ does not always exist for an arbitrary $\mathscr{B}$.

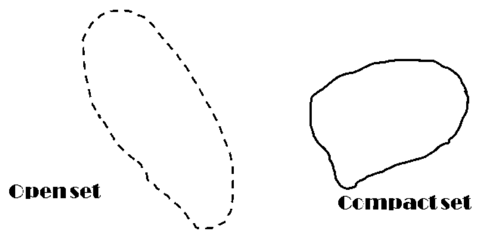

Figure 1.1: An open set and a compact set.

We review some important concepts in topological spaces:

- If O is an open set, then $\Omega \setminus O$ is called a closed set.
- The closure of $S \subset \Omega$ is the smallest closed set containing S.
- $S \subset \Omega$ is called compact if an arbitrary open cover of S has a finite subcover. Explicitly, for every arbitrary collection of open sets $\{U_\alpha\}_{\alpha \in I} \subset \mathscr{O}$ with I being the set of suffixes such that $S \subset \cup_{\alpha \in I} U_\alpha$, there exists a finite subset I' of I such that $S \subset \cup_{\alpha \in I'} U_\alpha$. In particular, if Ω is compact, then the topological space is called compact.

Example: In the usual topology in $\mathbb{R}^n$, a set $S \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

The followings are important properties that will be used in the proofs of the main argument.

- A closed subset of a compact space is compact.
- If $\{S_p\}_{p \in \mathbb{N}}$ be a family of compact subsets of a topological space such that $S_{p+1} \subset S_p$, then $\cap_p S_p \neq \phi$.

An important example of topological spaces is a Banach space.

A Banach spaces V is a complete normed vector space.
Here, a normed vector space is a vector space with a norm $\| \cdot \|$ satisfying:

- $\|x\| \geq 0$ for all $x \in V$, where $\|x\| = 0$ if and only if $x = 0$.
- $\|ax\| = |a|\|x\|$ with $a \in \mathbb{C}$.
- $\|x + y\| \leq \|x\| + \|y\|$.

A normed vector space is complete if every Cauchy sequence in V converges to an element of V. Explicitly, $\{x_n\}_{n \in \mathbb{N}} \subset V$ is called a Cauchy sequence, if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for every $n, m > N$.

$\mathbb{C}^n$ is a Banach space with the standard norm $\|x\| = \sum_{i=1}^{n} |x_i|^2$ with $x = (x_1, x_2, \cdots, x_n)$.

We next discuss continuous maps between two topological spaces Ω and Ω'.

Map $f; \Omega \to \Omega'$ is continuous if, for every open set $O' \subset \Omega'$, $f^{-1}(O')$ is an open set in Ω.

We can also define the continuity of map $f; \Omega \to \Omega'$ at a single point $x \in \Omega$. In usual $\mathbb{R}^n$, such a continuity can be defined in terms of the convergence of sequences. However, this definition is not enough in general. Instead, we need the concept of net.

A set is called net if it is labeled by a directed set I as $\{x_i\}_{i\in I}$. Here, I is a directed set if it has "$\le$" that satisfies the following properties:

- $a \le a$
- If $a \le b$ and $b \le c$, then $a \le c$.
- For arbitrary $a, b \in I$, there exists $c \in I$ such that $a \le c$ and $b \le c$.

We note that a sequence is a special net with $I = \mathbb{N}$.

A net $\{x_i\}_{i\in I} \subset \Omega$ converges to $x \in \Omega$ if, for every open set O with $x \in O$, there exists $i \in I$ such that $x_k \in O$ for all k with $i \le k$. We write this as $\lim_{i\in I} x_i = x$.

The following proposition is important.

$f; \Omega \to \Omega'$ is continuous at $x \in \Omega$ if $\lim_{i\in I} f(x_i) = f(x)$ holds for for every net $\{x_i\}_{i\in I}$ that converges to x.

$f; \Omega \to \Omega'$ is continuous if and only if it is continuous at all $a \in \Omega$.

We note that the concept of net is not necessary for first-countable spaces.[1] If the topological space is first-countable, every "net" above can be replaced by "sequence". Usual $\mathbb{R}^n$, every Hilbert space, and every Banach space are all first-countable in their norm topologies.

Finally, we discuss a way to create a topological space from another topological space.

Let Ω be a topological space and Ω' be its subset. Then we can define a topology on Ω' as follows: $O' \subset \Omega'$ is an open set if and only if there exists an open set $O \subset \Omega$ such that $O' = O \cap \Omega'$. This topology on Ω' is called relative topology.

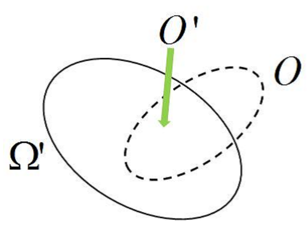

Figure 1.2: Relative topology.

[1]Here we only note that every metric space is first-countable.

Chapter 2

Hilbert Spaces and Operator Algebras

We shortly review some basic concepts of Hilbert spaces and operator algebras.

2.1 Hilbert Space

A Hilbert space $\mathscr{H}$ is a complex vector space with an inner product $(\cdot, \cdot)$ that is complete in terms of the norm $\|\varphi\|^2 \equiv (\varphi, \varphi)$. By definition, a Hilbert space is a Banach space.

$\mathbb{C}^n$ with the standard inner product is a n-dimensional Hilbert space and vice versa. We then generally introduce the orthonormal bases which can be applied even to "non-countable-dimensional" Hilbert spaces.

Let I be a set of suffixes. A set of vectors $\{\varphi_\alpha\}_{\alpha \in I} \subset \mathscr{H}$ is an orthonormal basis of $\mathscr{H}$, if it satisfies that:

- $(\varphi_i, \varphi_j) = \delta_{i,j}$.
- For any $\psi \in \mathscr{H}$, there exists a countable subset $I' \subset I$ such that

$$\lim_{N \to \infty} \|\psi - \sum_{n=1}^{N} (\varphi_{i_n}, \psi) \varphi_{i_n}\| = 0, \tag{2.1}$$

  where $\{i_1, i_2, \cdots\} = I'$.

Then we can show that:

Every Hilbert space has orthonormal bases, and their cardinalities are equal.

The above proposition leads to the dimension of Hilbert spaces.

Let $\{\varphi_i\}_{i \in I}$ be an orthonormal basis of $\mathscr{H}$. The cardinality of I is called the dimension of $\mathscr{H}$. If I is a countable set, then $\mathscr{H}$ is called separable.

In the following, we only consider separable Hilbert spaces.

A typical example of a separable infinite-dimensional Hilbert space is $L^2(\mathbb{R}^n)$, which is defined as the set of all Lebesgue measurable functions $f; \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} |f(x)|^2 dx < \infty. \tag{2.2}$$

In the case of infinite-dimensional Hilbert spaces, we can define two types of convergences. One is the ordinary convergence in terms of the standard norm, and the other topology is weaker than it.

(Strong) convergence: A sequence of points $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ is said to converge strongly (or simply "converge") to a point $\varphi \in \mathscr{H}$ if $\|\varphi_n - \varphi\| \to 0$.

Weak convergence: A sequence of points $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ is said to converge weakly to a point $\varphi \in \mathscr{H}$ if $(\psi, \varphi_n) \to (\psi, \varphi)$ for all $\psi \in \mathscr{H}$.

These two convergences are equivalent only in the case of a finite-dimensional Hilbert space. In general, if a sequence $\{\varphi_n\}$ strongly converges, then it weakly converges, because

$$|(\psi, \varphi) - (\psi, \varphi_n)| = |(\psi, \varphi - \varphi_n)| \leq \|\psi\|\|\varphi - \varphi_n\|. \tag{2.3}$$

However, the inverse of this does not always hold true for infinite-dimensional Hilbert spaces. For example, let us consider an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ of a separable Hilbert space. This sequence weakly converge to $0 \in \mathscr{H}$, because $\lim_{n\to\infty}(\varphi_n, \psi) = 0$ for all $\psi \in \mathscr{H}$. However, $\{\varphi_n\}$ does not converge to 0, because $\|\varphi_n - 0\| = 1$ for all n.

2.2 Bounded Operators

We will focus on operator algebras on a separable Hilbert space $\mathscr{H}$. An important class of linear operators on a Hilbert space is the set of bounded operators, which corresponds to the set of observables (and its linear combinations over $\mathbb{C}$).

A bounded operator x is a linear operator on $\mathscr{H}$ satisfying

$$\|x\| \equiv \sup_{\psi\in\mathscr{H}, \|\psi\|=1} \|x\psi\| < \infty, \tag{2.4}$$

where $\|\cdot\|$ is the norm of bounded operators, which actually satisfies the properties of norm. We write as $B(\mathscr{H})$ the set of all bounded operators on $\mathscr{H}$.

We next introduce the concept of adjoint.

It can be shown that for any $x \in B(\mathscr{H})$, there exists a unique operator $x^* \in B(\mathscr{H})$ such that

$$(x\psi, \phi) = (\psi, x^*\phi) \tag{2.5}$$

for all $\psi, \phi \in \mathscr{H}$.

- We call $x^* \in B(\mathscr{H})$ the adjoint of $x \in B(\mathscr{H})$.
- A bounded operator $x \in B(\mathscr{H})$ is self-adjoint if $x = x^*$.

The topology on $B(\mathscr{H})$ defined by norm $\| \cdot \|$ is called the uniform topology or the norm topology.

The following property plays an important role in the theory of operator algebras:

$B(\mathscr{H})$ is complete in terms of the norm $\| \cdot \|$, in other words, $B(\mathscr{H})$ is a Banach space.

Besides the uniform topology, $B(\mathscr{H})$ has several important topologies. They can be understood in terms of locally convex topologies that are defined in terms of seminorms.

A seminorm on a vector space V is a map $p; V \to \mathbb{R}$ satisfying:

- $p(x) \geq 0$ for all $x \in V$.
- $p(ax) = |a|p(x)$ with $a \in \mathbb{C}$.
- $p(x + y) \leq p(x) + p(y)$.

Note that if "$p(x) = 0$ if and only if $x = 0$ " is satisfied, the seminorm is a norm.

$\{p_\alpha\}_{\alpha \in I}$ is a collection of seminorms on a vector space V. The corresponding locally convex topology on V is the topology that is generated by the collection of subsets $\{U_{\alpha,\varepsilon}(x)\}_{\alpha,\varepsilon,x}$ of the form

$$U_{\alpha,\varepsilon}(x) \equiv \{y \in V \mid p_\alpha(y - x) < \varepsilon\}, \tag{2.6}$$

where $\alpha \in I$, $\varepsilon > 0$, and $x \in V$.

Obviously;

The uniform topology of $B(\mathscr{H})$ is the locally convex topology corresponding to the norm (that is also a seminorm) $\| \cdot \|$.

The following three locally convex topologies on $B(\mathscr{H})$ are crucial to understand operator algebras.

- The strong operator topology is the locally convex topology corresponding to seminorms $\{p_\varphi\}_{\varphi\in\mathscr{H}}$ where
$$p_\varphi(x) \equiv \|x\varphi\|. \tag{2.7}$$
- The weak operator topology is the locally convex topology corresponding to seminorms $\{p_{\varphi,\psi}\}_{\varphi,\psi\in\mathscr{H}}$ where
$$p_{\varphi,\psi}(x) \equiv |(\varphi, x\psi)|. \tag{2.8}$$
- Let $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, \cdots)$ ($\varphi_n \in \mathscr{H}$) satisfying $\sum_{n=1}^\infty \|\varphi_n\|^2 < \infty$, and $\boldsymbol{\psi} \equiv (\psi_1, \psi_2, \cdots)$ ($\psi_n \in \mathscr{H}$) satisfying $\sum_{n=1}^\infty \|\psi_n\|^2 < \infty$. The ultraweak topology or σ-weak topology is the locally convex topology corresponding to seminorms $\{p_{\boldsymbol{\varphi},\boldsymbol{\psi}}\}$ where
$$p_{\boldsymbol{\varphi},\boldsymbol{\psi}}(x) \equiv \left|\sum_{n=0}^\infty (\varphi_n, x\psi_n)\right|. \tag{2.9}$$

The upper two topologies can be understood in terms of the following two convergences.

Strong operator convergence: A net $\{x_i\}_{i\in I} \subset B(\mathscr{H})$ is said to converge strongly to a point $x \in B(\mathscr{H})$ if $\{x_i\psi\}_{i\in I} \subset \mathscr{H}$ strongly converges to $x\psi \in \mathscr{H}$ for all $\psi \in \mathscr{H}$.

Weak operator convergence: A net $\{x_i\}_{i\in I} \subset B(\mathscr{H})$ is said to converge weakly to a point $x \in B(\mathscr{H})$ if $\{x_i\psi\}_{i\in I} \subset \mathscr{H}$ weakly converges to $x\psi \in \mathscr{H}$ for all $\psi \in \mathscr{H}$.

We now have three convergences in $B(\mathscr{H})$. If a net[1] $\{x_n\}_{n\in\mathbb{N}} \subset B(\mathscr{H})$ uniformly converges, then it strongly converges; if it strongly converges, then it weakly converges.

A map f on $B(\mathscr{H})$ is continuous in terms of strong (weak) operator topology, if and only if f is continuous at all $x \in B(\mathscr{H})$ in terms of strong (weak) operator convergence.

We note that strong operator topology is stronger than weak operator topology. However, ultraweak topology is stronger than weak operator topology.

2.3 Trace Class Operators

Another important class of operator algebras is the trace class, which corresponds to the set of (unnormalized) density operators. To define it, we first note the concept of positivity.

- $x \in B(\mathscr{H})$ is positive, if $(x\psi, \psi) \geq 0$ for all $\psi \in \mathscr{H}$. Then $x = x^*$ holds.
- Let $x, y \in B(\mathscr{H})$. We write $x \leq y$ if $y - x$ is positive.

[1]Note that every norm topology is first countable.

For every $x \in B(\mathscr{H})$, there exists operator $y \in B(\mathscr{H})$ such that $y^2 = x^*x$. We write y as $|x|$. We can show that $|x|$ is positive for every x.

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of a separable Hilbert space $\mathscr{H}$, and $x \in B(\mathscr{H})$ be a positive operator. Then we define the trace of x as

$$\mathrm{Tr}(x) \equiv \sum_{n=1}^{\infty} (x\varphi_n, \varphi_n). \tag{2.10}$$

We can show that the definition of the trace is independent of the choice of the basis. We define the trace class $\mathscr{T}(\mathscr{H})$ as

$$\mathscr{T}(\mathscr{H}) \equiv \{x \in B(\mathscr{H}) \mid \mathrm{Tr}(|x|) < \infty\}. \tag{2.11}$$

The following properties are important.

- $\|x\|_1 \equiv \mathrm{Tr}(|x|)$ satisfies the properties of norms, which we call the trace norm.
- $\mathscr{T}(\mathscr{H})$ is complete in terms of the trace norm topology, which means that $\mathscr{T}(\mathscr{H})$ is a Banach space.

We next consider the concept of dual.

The dual of a normed vector space $\mathscr{V}$ is the set of continuous (in terms of the norm topology) linear maps from $\mathscr{V}$ to $\mathbb{R}$.

The following theorem is very important.

$B(\mathscr{H})$ is the dual of trace class $\mathscr{T}(\mathscr{H})$. The correspondence is given by

$$x(\rho) \equiv \mathrm{Tr}(\rho x), \tag{2.12}$$

where $x \in B(\mathscr{H})$ is interpreted as a functional on $\rho \in \mathscr{T}(\mathscr{H})$.

This theorem means that the map from "density operator" ρ to the expectation value of observable x is continuous in terms of the trace norm topology of $\mathscr{T}(\mathscr{H})$. On the other hand, according to the following theorem, the map from observable x to its expectation value with ρ is continuous in terms of the ultraweak topology of $B(\mathscr{H})$:

The ultraweak topology on $B(\mathscr{H})$ is the weakest topology with which all of the following maps are continuous:

$$\rho(x) \equiv \mathrm{Tr}(\rho x), \tag{2.13}$$

where $\rho \in \mathscr{T}(\mathscr{H})$ is interpreted as a functional on $x \in B(\mathscr{H})$.

Conversely, all states that are ultraweakly continuous can be expressed in terms of density operators:

A linear map $\Psi : B(\mathscr{H}) \to \mathbb{R}$ is called a state if it satisfies the following properties:

Positivity $\Psi(x^* x) \geq 0$ for all $x \in \mathscr{M}$.

Unity $\Psi(1) = 1$.

A state is called normal if it is ultraweakly continuous.

Every normal state Ψ corresponds to "density operator" $\rho \in \mathscr{T}(\mathscr{H})$ as

$$\Psi(x) = \mathrm{Tr}(\rho x), \tag{2.14}$$

where ρ is positive and $\mathrm{Tr}(\rho) = 1$.

Figure 2.1 summarizes the foregoing arguments.

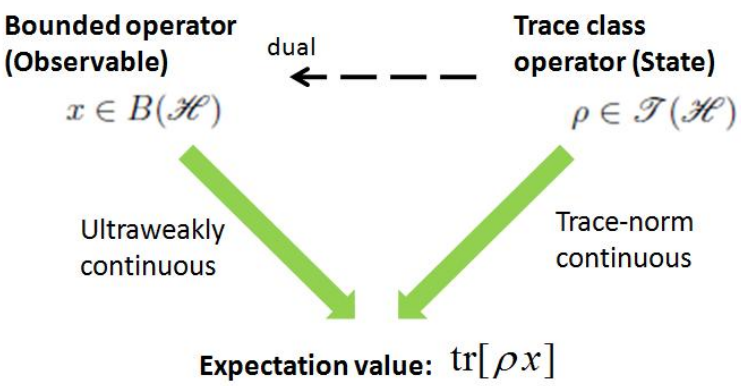

Figure 2.1: Illustration of the relationship between $\mathscr{T}(\mathscr{H})$ and $B(\mathscr{H})$.

2.4 von Neumann Algebras

We will next focus on important subalgebras of $B(\mathscr{H})$.

A $*$-subalgebra $\mathscr{M}$ of $B(\mathscr{H})$ is called a von Neumann algebra, if it has the identity 1 and is weakly closed. Here, $\mathscr{M}$ is a $*$-subalgebra if $x \in \mathscr{M} \Leftrightarrow x^* \in \mathscr{M}$.

Trivially, $B(\mathscr{H})$ itself is a von Neumann algebra. We note that we can define locally convex topologies on von Neumann algebras as the relative topologies corresponding to those of $B(\mathscr{H})$.

The von Neumann algebras can be characterized by some other ways. First of all,

Let $\mathscr{M}$ be a $*$-subalgebra of $B(\mathscr{H})$ containing the identity 1. Then the followings are equivalent:

- $\mathscr{M}$ is a von Neumann algebra.
- $\mathscr{M}$ is strongly closed.
- $\mathscr{M}$ is ultraweakly closed.

We next consider the double commutant.

Let $\mathscr{M}$ be an arbitrary subset of $\subset B(\mathscr{H})$. Then $\{a \in B(\mathscr{H});\ \forall x \in \mathscr{M}, ax = xa\} \subset B(\mathscr{H})$ is called commutant of $\mathscr{M}$, which is written as $\mathscr{M}'$.

von Neumann's bicommutant theorem: Let $\mathscr{M}$ be a $*$-subalgebra of $B(\mathscr{H})$ including the unity. Then $\mathscr{M}''$ is the closure of $\mathscr{M}$ in terms of, equivalently, strong, ultraweak, and weak topologies.

Consequently,

Let $\mathscr{M}$ be a $*$-subalgebra of $B(\mathscr{H})$ containing the identity 1. Then $\mathscr{M}$ is a von Neumann algebra if and only if $\mathscr{M} = \mathscr{M}''$.

We next discuss the concept of hyperfinite factors.

Let $\mathscr{M}$ be a von Neumann algebra. $\{a \in \mathscr{M};\ \forall x \in \mathscr{M}, ax = xa\} \subset \mathscr{M}$ is called center of $\mathscr{M}$.

$\mathscr{M}$ is called factor, if its center is given by $\{\alpha I; \alpha \in \mathbb{C}\}$ with I being the identity operator.

It is known that

Every finite-dimensional factor is isomorphic to $B(\mathbb{C}^n)$.

Factor $\mathscr{M}$ is called hyperfinite, if there exists a sequence of finite $n(p)$-dimensional factors $\{\mathscr{M}_p\}_{p\in\mathbb{N}}$ such that $\mathscr{M}_p \subset \mathscr{M}_{p+1}$ and $\mathscr{M} = (\cup_p \mathscr{M}_p)''$.

It is also know that

$B(\mathscr{H})$ with $\mathscr{H}$ being separable is hyperfinite.

2.5 Maps on von Neumann Algebras

We next consider maps from von Neumann $\mathscr{M}$ algebra to von Neumann algebra $\mathscr{N}$.

A linear map $T : \mathscr{M} \to \mathscr{N}$ is positive, if $Tx \geq 0$ holds for any positive operator $x \in \mathscr{M}$.

Let 1_n be the identity operator on $\mathbb{C}$. A positive map $T : \mathscr{M} \to \mathscr{N}$ is completely positive, if $T \otimes 1_n : \mathscr{M} \otimes \mathbb{C}^n \to \mathscr{N} \otimes \mathbb{C}^n$ is positive for all $n \in \mathbb{N}$.

We also define the norm of such maps:

A positive map $T : \mathscr{M} \to \mathscr{N}$ is bounded if

$$\|T\| \equiv \sup_{x\in\mathscr{M}, \|x\|=1} \|Tx\| < \infty, \tag{2.15}$$

where $\|T\|$ is called the norm of T.

Chapter 3

Abstract Operator Algebras

Operator algebras can be characterized without using Hilbert spaces.

3.1 C^*-Algebras

We first discuss C^*-Algebras.

$\mathscr{U}$, a vector space over $\mathbb{C}$, is called a C^*-Algebra if it satisfies:

1. A product is defined, which satisfies:
   - There is the unity.
   - $(xy)z = x(yz)$, where $x, y, z \in \mathscr{U}$.
   - $x(y+z) = xy + xz$ and $(x+y)z = xz; yz$, where $x, y, z \in \mathscr{U}$.
   - $(ax)y = x(ay) = a(xy)$, where $x, y \in \mathscr{U}$ and $a \in \mathbb{C}$.
2. An involution $* \,; x \in \mathscr{U} \mapsto x^* \in \mathscr{U}$ is defined, which satisfies:
   - $(c_1 x_1 + c_2 x_2)^* = c_1^* x_1^* + c_2^* x_2^*$, where $c_1, c_2 \in \mathbb{C}$ and $x_1, x_2 \in \mathscr{A}$.
   - $(x_1 x_2)^* = x_2^* x_1^*$.
   - $(x^*)^* = x$.
3. A norm $\|\cdot\| \,; x \in \mathscr{U} \mapsto \|x\| \in \mathbb{R}$ is defined, which satisfies:
   - $\|x\| \geq 0$.
   - $\|x\| = 0$ if and only if $x = 0$.
   - $\|cx\| = |c|\|x\|$ where $c \in \mathbb{C}$.
   - $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$.
   - $\|x^* x\| = \|x\|^2$.
4. $\mathscr{U}$ is complete in terms of the norm topology.

Clearly, $B(\mathscr{H})$ satisfies all of the above properties. Conversely, Every C^*-algebra is isomorphic to a norm-closed $*$-subalgebra of $B(\mathscr{H})$ with some $\mathscr{H}$.

We note that the following properties can be checked from the axioms of C^*-algebra:

- $\|1\| = 1$,
- $\|x^*\| = \|x\|$,
- $\|x_1 x_2\| \leq \|x_1\| \|x_2\|$.

3.2 W^*-algebras

Before we introduce W^*-algebra, we state another characterization of von Neumann algebras.

For every von Neumann $\mathscr{M}$, there exists a unique Banach space whose dual is $\mathscr{M}$.

This Banach space is called predual of $\mathscr{M}$. As we discussed, the predual of $B(\mathscr{H})$ is $\mathscr{T}(\mathscr{H})$. Based on this property,

A C^*-algebra $\mathscr{M}$ is called a W^*-algebra, if there is a Banach space such that its dual is $\mathscr{M}$. We note that the Banach space is uniquely defined.

Every W^*-algebra is isomorphic to a von Neumann algebra.

Therefore, we can identify a W^*-algebra to a von Neumann algebra.